Principle Bundles and Connections

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Outline

1. Introduction
   - Motivation
   - Basic Definitions
2. Fibre Bundles
3. Principal Bundles
4. Horizontal Distributions
5. Principal Connections
   - horizontal Distribution of an ehrensmann connection
6. Frame Bundle:Connections
7. Assosciated Bundles

Why Fiber Bundles?

Recall: Last Week we saw Vector Bundles over Manifolds.

Questions:

- Can we generalize this so that each Fiber is a topological Space or more precisely a Manifold?
- What kind of Spaces can we construct and what new properties do they have compared to Vector Bundles ?
- What happens if we let the Fibers be a Lie Group?

In this Talk we will see that these Spaces can be constructed and allow us to understand familiar manifolds better. We will mostly follow Loring W. Tu, Differential Geometry, Chapters 27-30

Lie Group Actions on Manifolds

G will usually denote a Lie Group in this Talk unless stated otherwise.(see John m. Lee, Differential Geometry for more)

right G-actions

A smooth right G-action on a manifold is given by a smooth map$f : M \times G \to M$ given by $f(p, g) \to pg$ such that $\forall p \in M$ and $\forall g, h \in G$ the following hold

- $xe = x$, where e is the identity of G
- $(xg)h = x(gh)$

The set $Gp = \{pg | g \in G\}$ is called the Orbit of p for some $p \in M$
The set $G_p = \{pg = p | g \in G\}$ is called the stabilizer of p.

G-actions continued

Some Remarks:

- Similarily to right actions one can also define left actions.
- given a left action, taking $pg = g^{-1}p$ defines a right action(and vice versa).
- a Group action is said to be free if Stab(x) for every $x \in M$ is the trivial subgoup $\{e\}$
- for each $x \in M$ the Mapping: $G \to xG$,$g \to xg$ induces a bijection:

$$Stab(x)\backslash G \leftrightarrow xG$$

$$Stab(x)g \leftrightarrow xg$$

G-equivariance

right and left G-equivariance

A Manifold together with a right(or left) G-Action is called right(or left) G-Manifold. If N, M are right G-Manifolds a map $f : N \to M$ is said to be right G-equivariant if

- $f(xg) = f(x)g$

since we can define from a left action a right action it is sensible to consider:

G-equivariant

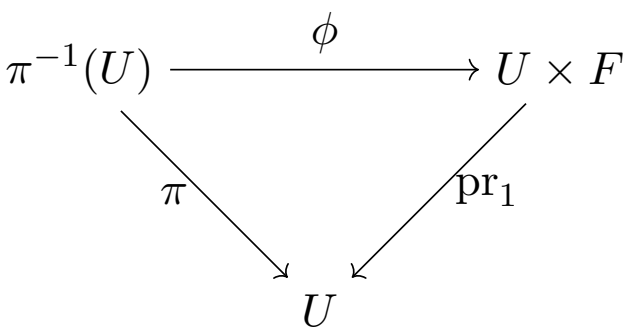
If N is a right G manifold and M is a left G Manifold then a map $f : N \to M$ is said to be G-equivariant if

- $f(xg) = f(x)g = g^{-1}f(x)$ for all $(x, g) \in N \times G$

Fibre bundles

Definition

Let E, M and F be (smooth) manifolds and $\pi : E \to M$ be a surjective smooth map. Then the pair(E,π,M,F) is called a (locally trivial) fiber bundle if for each $p \in M$ there is an open set U containing p and a smooth Diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the Diagram below commutes



Fiber Bundles

On a Fiber Bundle (E, π, M, F) we use following therminology

- E is called the Base Space
- F is called a typical fiber
- for each $p \in M$ the set $E_p = \pi^{-1}(p)$ is called a fiber over p
- one often refers to the Bundle by the map π

Example

A simple example of a fiber bundle is is the following: Given two manifolds M, F and the map $\mathrm{pr}_1 : M \times F \to M$ the Tuple $(M \times F, \mathrm{pr}_1, M, F)$ is then called the trivial bundle.

More Examples and A Proposition

- Vector bundles are Fiber Bundles!
- The Hopf Bundle(covered in due time)
- The Frame Bundle which will be covered in detail later, is a $GL(r, \mathbb{R})$-bundle over a Vector Bundle.
- The last two will be a more special type of Bundle called Principle Bundles.

Proposition

Let $\pi : E \to M$ be a fiber bundle with Fiber F: If
$\phi_U : \pi^{-1}(U) \to U \times F$ is a trivialisation, thn
$\phi_{U|E_x} : E_x \to \{x\} \times F$ is a diffeomorphism.

Proof and Definition

Proof: since ϕ_U is smooth so is it's restriction to the Fiber E_x. Similarly the inverse $\phi^{-1}_{U|E_x} : \{x\} \times F \to E_x$ is therestriction of $\phi^{-1}_U : U \times F \to \pi^{-1}(U)$ to $\{x\} \times F$Ṡince ϕ_U is a bijection so is it's restriction.

Principal Bundles

A smooth fiber bundle $\pi : P \to M$ of Manifolds M, P with a Lie Group G as fiber, such that the right action of G on P is free and the local trivialisations $\phi_U : \pi^{-1}(U) \to U \times G$ are G-equivariant, with the action of G on $U \times G$ on the right given by (x,h)·g = (x, hg) is called $Principal\ G\text{-}Bundle$

Hopf Bundle

before proving the proposition we introduce:

Hopf Bundle

The group S^1 can act on $\mathbb{C}^{n+1}$ by left multiplication. By viewing $S^{2n+1} \subset \mathbb{C}^{n+1}$. Since we can then map $S^{2n+1} \to \mathbb{C}P^n$ via the natural projection. With Fibers S^1 this becomes a principal S^1 bundle. When n=1 this is then the **Hopf bundle**

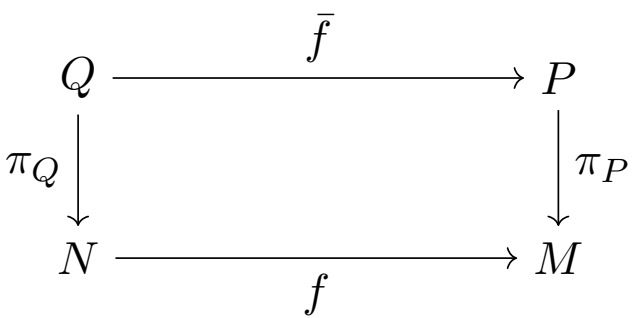
Remark:

- The Hopf Bundle plays a role in the Theory of magnetic Monopoles which is explored in: "The Geometry of Physics" by Theodore Frankel

Principal Bundle Mophisms

G-Bundle Morphism

Let $\pi_Q : Q \to N$ and $\pi_P : P \to M$ be principal G-Bundles. A morphism of principal G-bundels is a pair of maps: $\bar{f} : Q \to P$,$f : N \to M$) such that $\bar{f}$ is G-equivariant and the diagram below commutes.



G- equivariance is left translation

Recall that on a map the maps $L_g : G \to G$ and $R_g : G \to G$ are called left and right translation and both these maps are diffeomorphisms.

Lemma

For any group G a right G-equivariant map $f : G \to G$ is necessarily a left translation.

proof:
Suppose for all $x, g \in G$ the following holds

$$f(gx) = f(x)g$$

Letting $x = e$ it follows

$$f(g) = f(e) = L_{f(e)}(g)$$

Transition Functions

Suppose $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ is a trivialization for a principal G-Bundle $\pi : P \to M$. For $U_{\alpha\beta} = U_\alpha \cap U\beta \neq \emptyset$, we have two trivializations on $\pi^{-1}(U_{\alpha\beta})$, namely ϕ_α and ϕ_β
$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ is a fiber preserving right G equivariant map. By the previous lemma and by the definition of the right action this mao is a Left Translation on each fiber. i.e:

$$\phi_\alpha \circ \phi_\beta^{-1}(x, h) = (x, g_{\alpha\beta}(x)h) \tag{1}$$

The map $g_{\alpha\beta} : U_{\alpha\beta} \to G$ is a smooth function(to see this recall that ϕ_α are diffeomorphisms and set $h = e$) called transiotion function of the principal bundle $\pi : P \to M$ relative to $\{U_\alpha\}_{\alpha \in A}$

Transition Functions

for $\alpha, \beta, \gamma \in A$ the transition functions satisfy:

cocycle condition

$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ if $U_\alpha \cap U_\beta \cap U_\gamma$ is non empty

which can be seen by inserting $id_{\pi^{-1}(U)} = \phi_\gamma^{-1} \circ \phi_\gamma$ into (1) . from this we get:

Proposition

on a principal bundle $\pi : P \to M$ relative to a trivializing cover $\{U_\alpha\}_{\alpha \in A}$ satisfy the following properties: for all $\alpha, \beta \in A$

- (i) $g_{\alpha\alpha} = e$
- (ii) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ if $U_\alpha \cap U_\beta$ is nonempty

proof: for (i) take $\alpha = \beta = \gamma$ and for (ii) take $\gamma = \alpha$ in the cocycle condition and use (i) □

Remarks and Frames on Vector Spaces

We now make some Observations

- Note that from a right action of the Group on the total space we get transitionfunctions which are given by Left Translation as a direct consequence of Lemma 1.
- Last week we introduced $Fr(V)$ as the set of all ordered Basis on a Vector Space of dimension r and studied this set on Vector Bundles.
- We also saw how the Group $GL(r, \mathbb{R})$ acts on $FR(V)$.

It is thus natural to recontextualize these results in the Language of Principal Bundles. furthermore we denote an ordered Basis $v_1, ..., v_r$ of V by $v = [v_1, .., v_r]$ and the right action of $GL(r, \mathbb{R})$ by $v \cdot a = [\sum v_i a_1^i, ..., \sum v_i a_r^i]$

Frame Manifold

by fixing a point $v \in Fr(V)$ and by virtue of the fact that the action of $GL(r, \mathbb{R})$ on $Fr(V)$ is free and transitive we get from the orbit Stabilizer Theorem(without proof) a bijection given by

$$\phi_v : GL(r, \mathbb{R}) \leftrightarrow Fr(V)$$

$$g \leftrightarrow vg$$

Then we put a manifold structure on Fr(v) such that ϕ_v becomes a diffeomorphism
if $v' = va$ is another Frame then they define The same Manifold structure since: $\phi_{va}(g) = vag = (\phi_v \circ L_a)(g)$

Frame Manifold

Fr(V) together with the construction above is then called the Frame Manifold

Frame Bundle

Remark without proof

Given two finite dimensional Vector Spaces V,W and a linear Isomorphism $\phi : V \to W$ we get an induced Diffeomorphism $\phi' : Fr(V) \to Fr(W)$ by:

$$\phi' v = [\phi(v_1), ..., \phi(v_r)]$$

Frame Bundle

Let $\eta : E \to M$ be a smooth vector bundle of rank r (i.e A Fiber Bundle with an r Dimensional Vector space as Fibers). We associate to the vector bundle a smooth principal $GL(r, \mathbb{R})$ bundle $\pi : Fr(E) \to M$ called the Frame Bundle via the following construction:

Continuation/Construction of the Frame Bundle

We take the total space as $Fr(E) = \bigcup_{x \in M} Fr(E_x)$ and let $\pi : Fr(E) \to M$ be the natural projection $Fr(E_x) \to \{x\}$
A local trivialisation $\phi'_\alpha : E_{|U_\alpha} \mapsto U_\alpha \times Fr(\mathbb{R}^r)$ via

$$v \in Fr(E_x) \mapsto (x, [\phi_{\alpha,x}(v_1), .., \phi_{\alpha,x}(v_r)])$$

By ϕ'_α we can transfer the topology and manifold structure from $U_\alpha \times Fr(\mathbb{R}^r)$ to $Fr(E)|_{U_\alpha}$. In this manifold structure $\pi : Fr(E) \to M$ is locally trivial with Fiber $Fr(\mathbb{R}^r)$
Because the Frame Manifold $Fr(\mathbb{R}^r)$ is diffeomorohic to $GL(r, \mathbb{R})$ we then have that $Fr(E) \to M$ is A C^∞ principal $GL(r, \mathbb{R})$ Bundle

Transition Functions on The Frame Bundle

we now turn our attention to the previously mentioned Transition functions.
On a nonempty overlap $U_{\alpha\beta}$ the transition function for the vector Bundle E is $g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R})$ given by:

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^r \to U_{\alpha\beta} \times \mathbb{R}^r$$

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, w) = (x, g_{\alpha\beta}(x)w)$$

Since the trivialisation For Fr(E) is induced by the trivialisations on E, so are the transition functions $g_{\alpha\beta}$ and by the Remark on linear isomorphisms they are the Same, only now acting on $Fr(\mathbb{R}^r)$ instead of $\mathbb{R}^r$

Horizontal Distributions

To define Connections on a Principal Bundle we need to first study horizontal Distributions on principle Bundles

Vertical Bundles

Let P be the total Space $\pi : P \to M$ a principal G-Bundle denote by TP the tangent bundle. The we call $\mathcal{V}_p = ker(\pi_{*,p})$ the Vertical subundle at p. a tangent Vector $v_p \in \mathcal{V}_p$ is called vertical tangent Vector at p. Furthermore we have $dim(\mathcal{V}_p) = dim(G)$

Horizontal Distribution

let P be the total Space and let $\pi : P \to M$ be a principal G bundle. By $\mathcal{V}$ we denote the Vertical Subbundle of TP. We call a subbundle $\mathcal{H}$ of TP horizontal if $TP = \mathcal{V} \oplus \mathcal{H}$ as Vector Bundles(i.e $T_pP = \mathcal{V}_p + \mathcal{H}_p$ and $\mathcal{V}_p \cap \mathcal{H}_p$)

Splitting of Vector Bundles

Remark:

- in general there is not a canonical way of defining a horizontal Distribution on a Principal bundle.

Splitting of a Sequence

A splitting of short exact sequences of Vector bundles $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a bundle map $k : C \to B$ s.t $j \circ k = id_C$

Proposition 3

let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ denotet by W be a short exact sequence of vector fields over a Manifold M. Then there is a bijection: $\{subbundles\ H \in B | B = i(A) \oplus H\} \leftrightarrow \{splittings\ k : C \to B\ of\ W\}$

Proof of the Proposition

If H is a subbundle of B s.t $B = i(A) \oplus H$ then there are bundle isomorphisms $H \simeq B/i(A) \simeq C$
hence C maps isomorphically onto H in B. This gives a slpitting $k : C \to B$.
If $k.C \to B$ is slpitting, let $H := k(C)$, which is a subbundle of B.
Additionally if for some $a \in A$ and $c \in C$
$i(a) = k(c) \Rightarrow 0 = ji(a) = jk(c) = 0$ this means $i(A) \cap k(C) = 0$
It remains to show $B = i(A) + k(C)$.
let $b \in B$ Then:

$$j(b - kj(b)) = j(b) - j(b) = 0$$

furthermore $b - kj(b) = i(a)$ for some $a \in A$ by the exactness of W. Therefore:

$$b = i(a) + kj(b) \in i(a) + k(C)$$

So in total $B = i(A) + k(C)$ and thus $i(A) \oplus k(C)$ □

exact sequence of vertical bundle

the differential $d\pi : TP \to TM$ inducesa bundle map $\tilde{\pi}_* : TP \to \pi^* TM$ given by $X_p \to (p, \pi_{*,p} X_p)$, which is surjectiv. the kernel is the vertical subbundle. we get the sequence

$$0 \to \mathcal{V} \to TP \to \pi^* TM \to 0$$

so we get a one to one mapping between horizontal distributions and splittings of this sequence

Remarks

Let G be a Lie group with lie Algebra $\mathfrak{g}$.

- on principal G-bundle $P \to M$ Vertical subbundles are well defined but not horizontal Distributions.
- we will see that Connections on a Principal Bundle are essentially a choice of a horizontal Distribution on P
- similar to the case on Vector Bundles a connection will be given by one forms.

we will see two equivalent types of Connections

1. a smooth right-invariant horizontal distribution on P
2. a smooth G equivariant $\mathfrak{g}$ valued 1 form ω on P such that on the fundamental Vector Fields $\omega(\underline{A}) = A,\ \forall A \in \mathfrak{g}$

Fundamental Vector fields.

To not exceed the scope of this talk we introduce fundamental Vector fields.(for more see Loring W. Tu, Differential Geometry Chapter 27)

- Let $\pi : P \to M$ be a principal G-bundle. if for $p \in P$ $j_p : G \to P$ is given by $j_p(g) = pg$ the Vertical tangent space $\mathcal{V}_p$ can canonically be identified with $\mathfrak{g}$ via the Isomorphism
$$j_{p*} : \mathfrak{g} \to \mathcal{V}_p$$
- a fundamental Vector Field associated to an element $A \in \mathfrak{g}$on P given by $\underline{A}_p = \frac{d}{dt}(pe^{At})(0) \in T_pP$

vertical components

let $\nu : T_pP = \mathcal{H}_p \oplus \mathcal{V}_p \to \mathcal{V}_p$ be th natural projection. then for $Y_p \in T_pP$ the image $\nu(Y_p)$ is called the vertical component of Y_p Important Remark: although $\mathcal{V}_p$ is intrinsic to P the notion of vertical components still depend on the choice of $\mathcal{H}_p$!

Principal Connections

Definition and Theorem

if for each $p \in P$ we take ω_p as

$$\omega_p := j_{p*}^{-1} \circ \nu : T_pP \to \mathfrak{g}$$

then ω is a $\mathfrak{g}$ valued 1-form on P sucht that the vertical components for $Y_p \in T_pP$ are $\nu(Y_p) = j_{p*}(\omega_p(Y_p))$ and if $\mathcal{H}$ is a smooth right invariant horizontal distributon on P of a principal G-bundle $\pi : p$ the following hold:

1. for any $A \in \mathfrak{g}$ and $p \in P$: $\omega_p(\underline{A}_p) = A_p$
2. (G-Equivariance) for any $g \in G$: $R_g^*\omega = (Adg^{-1})\omega$
3. ω is a smooth 1 form.

Proof of theorem

we now prove the assertions:

- (i) $\underline{A}_p$ is vertical, thus $\nu(\underline{A}_p) = A_p$. This means then:

$$\omega(\underline{A}_p) = j_{p*}^{-1}(\nu(\underline{A}_p)) = j_{p*}^{-1}(\underline{A}_p) = A$$

- (ii) for $p \in M$ and $Y_p \in T_pP$ we use that the equation we want to show is linear on both sides. Thus we treat the cases of Y_p being vertical and hoizontal seperately.
- Let Y_p be vertical then $Y_p = \underline{A}_p$ for some $A \in \mathfrak{g}$ then by (i) and because j_{*p} is an isomorphism we evaluate the right side of (ii) acting on $Y_p = \underline{A}_p$ for some A at p and get:

$$\omega_{pg}(r_{p*}\underline{A}_p) = \omega_{pg}(\underline{(Adg^{-1})A}_{pg}) = (Adg^{-1}))\omega_p(\underline{A}_p)$$

continuation of the Proof

- the horizontal case follows by right invariance and the definition of ω
- (iii) fix $p \in P$. Let $\{B_i\}$ where $i = 1, .., l$ be a basis of $\mathfrak{g}$ and $\{\underline{B_i}\}$ be the associated fundamental fields on P. these are all smooth
  Since $\mathcal{H}$ is a smooth distribution on P there is a Nbhd. W of p and smooth horizontal V. Fields $X_1, ..., X_n$ on W that span $\mathcal{H}$ at every $q \in W$.
- $\underline{B}_1, ..., \underline{B}_l, X_1, ..., X_n$ is a smooth frame of TP over W. Furthermore A smooth VF on W can be written as $X = a^i \underline{B}_i + b^j X_j$ for some smooth coefficent functions a^j, b^j now ω becomes:

$$\omega(X) = \omega(a^i \underline{B}_i) = a^i B_i$$

and is thus smooth



Ehrensman Connection

We can also much more simply define a connection by:

(Ehrensmann) Connection of a Principal Bundle

let $P \to M$ be a principal G bundle. then an Ehrensman Connection is a $\mathfrak{g}$ valued 1 form satisfying the Properties of the Previous Theorem.

- a 1 form values in $\mathfrak{g}$ above is a map $\alpha : TP \to \mathfrak{g}$ both of these sets a are G-manifolds(G acting by R_{g*} on TP or by the the adjoint representation on $\mathfrak{g}$)
- by a previous result α is G-equivariant iff $\forall p \in P, X_p \in T_pP$ and $g \in G$

$$\alpha(X_p g) = g^{-1}\alpha(X_p)$$

Equivalently:

$$\alpha(R_{g*}X_p) = Adg^{-1}\alpha(X_p)$$

Tangent Vectors: Vertical and Horizontal Components

So α is G-equivariant (Condition (ii) of a connection) iff $R_g^* \alpha = Adg^{-1}\alpha$ for all g in G

vertical and horizontal decompostion

Given a Hoizontal Distribution on P of a principal Bundle, every tangent vector $Y_p \in T_pP$ decomposes via $Y_p = \nu(Y_p) + h(Y_p) \in \mathcal{V}_p \oplus \mathcal{H}_p$. which are called vertical/horizontal component of Y_p. this extends to a vector field as $Y = \nu(Y) + h(Y)$

Proposition 4

if $\mathcal{H}$ is a smooth horizontal Distribution on P of a principal bundle, then the vertical and horizontal components of a vector field Y on P are also smooth.

Proof

let ω be a $\mathfrak{g}$ valued 1-form asscociated to $\mathcal{H}$ by the previous theorem it's smooth. In given a basis Basis $B_1, ..., B_l$ of the lie Algebra $\omega = \omega^i B_i$ where the coefficients are smooth real valued 1-forms.
Say $Y_p \in T_pP$ is a tangent Vector. then the vertical component is

$$\nu(Y_p) = j_{p*}(\omega_p(Y_p)) = j_{p*}(\omega_p^i(Y_p)B_i) = (\omega_p^i(Y_p))(\underline{B}_i)$$

So for any p: $\nu(Y) = \omega^i(Y)\underline{B}i$ since these are all smooth objects so is $\nu(Y)$ and thus $h(Y) = Y - \nu(Y)$ is also smooth $\square$

horizontal Distribution of an ehrensmann connection

- We saw how a horizontal Distribution determines a connection
- We therefore wonder if the converse is also true, which will turn out to be the case

Theorem

If ω is a connection on a Principal G-bundle $\pi : P \to M$ then $\mathcal{H} := ker(\omega_p)$ is a smooth right invariant horizontal distribution.

The proof is structured in Steps:

- show that in each p in P, T_pP decomposes as $\mathcal{V}_p \oplus \mathcal{H}_p$
- For p in P and g in G show that $R_{g*}(\mathcal{H}_p) \subset \mathcal{H}_{pg}$
- $\mathcal{H}$ is a smooth Subbundle of TP

We will not cover the proof(it can be found in Loring W. TU Chapter 28)

Horizontal lift

before describing Connections on the Frame bundle we need a notion of Horizontal lift, or a way to take Vector Fields on M to the total Space P. because $\ker(\pi_{p*})$ is $\mathcal{V}_p$ we get an isomorphisms:

$$\frac{T_pP}{ker(\pi_*)} \to \mathcal{H}_p \to T_{\pi(p)}M$$

Definition and Proposition 5

Let P be the total Space of $\pi : P \to M$ and $X \in \Gamma(TM)$.
the horizontal vector $\tilde{X}_p \in \mathcal{H}_p$ is uniquely given by $\pi_*(\tilde{X}_p) = X_{\pi(p)} \in T_{\pi(p)}M$ and the corresponding Vector Field is given by $\tilde{X}$ is called horizontal lift of X to P.
If H is a smooth right-invariant Horizontal distribution on P of a Principal Bundle $\pi : P \to M$ then $\tilde{X} \in \Gamma(TP)$ and $\tilde{X}$ is right invariant

Proof of Proposition

- Let $x \in M, p \in \pi^{-1}(x)$. By definition$d\pi(\tilde{X}_p) = X_x$ now let $p \neq q \in p \in \pi^{-1}(x)$ then $q = pg$ for some g in G. thus $d\pi(dR_g\tilde{X}_p) = d(\pi \circ R_g)\tilde{X}_p = X_p$
- by uniqueness we get then $dR_g\tilde{X}_p = \tilde{X}_{pg}$ so we have that the horizontal lift is right invariant.
- to prove smoothness we take an open cover $\{U\}$ of P that is trivializing with $\phi_U : \pi^{-1}(U) \to U \times G$ and let $Z := (X_x, 0) \in T_{(x,g)}(U \times G))$
  If $\tau : U \times G \to U$ is the projection then $d\tau_{x,g} = X_x$ and $Y := d\phi_U^{-1}(Z)$ is in $\Gamma(T\pi^{-1}(U))$ s.t $d\pi Y_p = X_{\pi(p)}$
- It follows then that $h(Y)$ is a smooth vector field on$\pi^{-1}(U)$ by the decomposition $Y_p = \nu(Y_p) + h(Y_p)$ we have:

$$\pi_*(Y_p) = \pi_* h(Y_p) = X_{\pi(P)}$$

- continued on next slide

Proof and short Lemma

- then by the uniqueness we conclude $h(Y) = \tilde{X}$ over U. Thus the horizontal lift is in $\Gamma(TP)$ □

We state without proof:

Lemma 2

Let P is a principal bundle with a connection. then for the fundamental VF of $A \in \mathfrak{g}$ then he following hold:

- if Y is hoizontal then so is $[\underline{A}, Y]$
- if Y is a right-invariant VF on P then $[Y, \underline{A}] = 0$

parallel frames

We now take the Frame Bundle as an example and illustrate how it is an example of what we have constructed so far.
recall:

- last week we defined parrallel sections of Vector Bundles and saw that they must satisfy a System of ODE's.
- we concluded, that provided enough information there is a always a parrallel section along a curve for some short time.
- using this we defined parralell transport of sections alng a path and showed that it is an isomorphism.

Thus we can define:

parallel frame

parallel frame a frame along a curve$c : [a, b] \to M$ is a basis $\{e_1(t), .., e_n(t)\}$ of $E_{c(t)}$ for each $t \in [a, b]$ that is parallel.

horizontal lift of frames

lift of curves

if $\pi : Fr(E) \to M$ denotes the frame bundle of $\eta : E \to M$ then a curve $\tilde{c}(t) : [a, b] \to Fr(E)$ is called lift of c(t) if $\pi(\tilde{c}(t)) = c(t)$.
if in addition $\tilde{c}(t)$ is parralel c along it is called a horizontal lift.

since parallel transport is an isomorphism if at some t_o there exists a frame parallel to c, then for some short time there exists a unique horizontal lift of that frame.
Specifically on frame bundles we define:

Horizontal Vectors on a frame bundle.

Let $E \to M$ be a vector Bundle with a connection∇. For x in M and $e_x \in Fr(E_x)$. a tangent Vector $v \in T_{e_x} Fr(E_x)$ is called horizontal if there is a curve c(t) through x in M such that is the initial vector $\tilde{c}'(0)$ of the unique horizontal lift $\tilde{c}(t)$ of c(t) is precisely that vector.

Horizontal Lifts: expression for $\tilde{c}(t)$

We want to get an expression of $\tilde{c}'(t)$. In terms of a local frame for E. we this in the following way:

- let $c : [0, a] \to M$ is a smooth curve with initial point $x = c_0$, denote by $\tilde{c(t)}$ the unique horizontal lift to Fr(R) with initial point $e_x = (e_{1,0}, ..., e_{r,0})$
- Take now s to be a frame over a neighbourhood U of x. with $s(x) = e_x$ so $s \circ c(t)$ is a (a priori) not horizontal lift
- but since we have for each t two ordered frames of $E_{c(t)}$ so they are connectet for each t by a matrix $a(t) \in GL(r, \mathbb{R})$ $a(0) = I$

Lemma 3

In the Notation above, let $s_* : T_xM \to T_{e_x}(Fr(E)$ be the diferential of s. Denote by $\tilde{a}'(0)$ the fundamental vector field on Fr(E) associated to $a'(0) \in \mathfrak{gl}(r, \mathbb{R})$ then.

$$s_*(c'(0)) = \tilde{c}' + \underline{a}'(0)$$

Proof:

- let $\mu : Fr(E) \times G \to Fr(E)$ be the right action of G on $Fr(E)$. We have $s(c(t) = \tilde{c}(t)a(t) = \mu(\tilde{c}(t), a(t))$ with a,s,c defined as above.
- we differentiate at 0 and get:

$$s_*(c(0) = \mu_{*,(\tilde{c}(0),a(0))}(\tilde{c}'(0), a(0))$$

- we use without proof the formula for the differential of an action to get

$$s_*(c'(0)) = r_{a(0)*}c'(0) + \underline{a'(0)} = \tilde{c}'(0) + \underline{a}'(0)_{e_x}$$

□

Lemma 4

Let $E \to M$ be a vector bundle of rank r with a connection ∇. Let s be a frame for E over an open set U, $\tilde{c}(t)$ a parallel frame over a curve $c(t)$ with image in U, s.t $\tilde{c}(0) = s(c(0))$. Let $a(t)$ be a curve in $GL(r, \mathbb{R})$ s.t $s(c(t)) = \tilde{c}(t)a(t)$. If now by $\omega_s = [\omega]^i_j$ we denote the connection matrix wrt. s over U, then $a'(0) = \omega_s(c'(0))$

Proof:

- we start by labeling $x = c(0)$ and $\tilde{c}_i(0) = s_i(c(0)) = e_{i,x}$ and obtain by the definition of the connection:

$$\nabla_{c'(0)} s_j = \omega^i_j(c'(0))s_i(c(0)) = \omega^i_j(c'(0))e_{i,x}$$

Proof of Lemma and a Proposition

- also $\nabla_{c'(0)} s_j = \frac{D(s_j \circ c)}{dt}(t) = \frac{D}{dt}\tilde{c}_i(t) a^i_j(t) = a^{i'}_j(t)\tilde{c}_i(t)$ since $\tilde{c}_i$ is parallel. and at t=0

$$\nabla_{c'(0)} s_j = a^{i'}_j(0)\tilde{c}_i(0)$$

□

Proposition 20

Suppose $\pi : E \to M$ is a smooth vector bundle with connection over M^n. For $x \in M$, e_x a ordered basis for E_x, the Subset $\mathcal{H}_{e_x} \subseteq T_{e_x} Fr(E)$ of horizontal vectors is a vector space of dimension n and $\pi_* : \mathcal{H}_{e_x} \to T_x M$ is a linear isomorphism.

sketch of proof: rewrite the formula proved in lemma 4 with ω_s and use that the of assigning fundamental vector fields is $\mathbb{R}$-linear. Then use that $c'(0) \to \tilde{c}'(0)$ is linear.
differntiate $\pi\tilde{c}(t)$ at 0 and show that this is an inverse. □

Horizontal lifts to a Frame bundle

We can decompose $T_p(Fr(E)) = \mathcal{V}_p \oplus \mathcal{H}_p$ on the frame bundle of a Vector Field
we want to make a smooth horizontal distribution $\mathcal{H}_p := \cup_{p \in P} \mathcal{H}_p$ of TFr(E)
we use that π_* is a linear isomorphism on $\mathcal{H}_p$ for each p. thus if $X \in \Gamma(TM)$ there is a unique Vector Field $\tilde{X}$ related to X via: $\pi_{*,P}(\tilde{X}_p))$

horizontal lift formula

Let ∇ be a connection on a vector bundle $E \to M$ and ω_s it's connection matrx on an open set U wit frame s. for $x \in U$, $p = s(x) \in Fr(E)$ and $X_x \in T_xM$. Let $\tilde{X}_p$ be the horizontal lift of X then

$$\tilde{X}_p = s_{*,x}(X_x) - \omega_s(X_x)$$

Horizontal distribution induced by a connection

We prove the Formula and proceed without showing that the Horizontal Lift defines a Vector Field.

- Choose for each $X_x \in T_xM$ a curve $c(t)$ s.t $c'(0) = X_x$ and rewrite with Lemma 5 and 4 □

We Can now state a concluding Theorem:

Theorem

If ∇ is a connection on a smooth vector bundle $E \to M$, then it defines A smooth horizontal distribution $\mathcal{H}$ on the Frame bundle $\pi : Fr(E) \to M$ satisfying the following at each $p \in P = Fr(E)$

1. $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$
2. $r_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ for all $g \in GL(r, \mathbb{R})$

Proof of Theorem

proof:

- First let U be an coordinate neighbourhood of M and $s_1, ..., s_n$ a smooth frame on U. The horizontal lifts are then smooth vector fields $\tilde{s}_1, ..., \tilde{s}_n$ on $\tilde{U} := \pi^{-1}(U)$.
- for each $p \in \tilde{U}$ the lifted frame forms a basis of $\mathcal{H}_p$ since π_{*p} is an isomorphism of Vector Spaces.
- then on $\tilde{U}$ $\tilde{s}_1, ..., \tilde{s}_n$ span $\mathcal{H}$ and thus $\mathcal{H}$, is a smooth Subbundle of TP via Theorem 20.4 from (Woring Tu chapter 20)

we then continue by showing the properties 1 and 2 on the next slide.

Continuation of Proof

- The first property follows directly by $\mathcal{V}_p = ker(\pi_{*p})$ for each p
- For the second Property let $c : [a, b] \to M$ be a curve and $\tilde{c}(t) = [v_1, ..., v_n]$ be the horizontal lift. furthermore choose the curve such that $\tilde{c}'(0) \in \mathcal{H}_p$. for $g \in GL(r, \mathbb{R})$ the group action on Fr(E) is matrix multiplication at each point.

$$\tilde{c}(t)g = [g_1^j v_j(t), ..., g_r^j v_j(t)]$$

because v_j are horziontal and g is constant $\tilde{c}(t)g$ is the horizontal lift with inital point $\tilde{c}(0)g$

- it therefore follows $\frac{d}{dt}|_{t=0}(\tilde{c}(t)g) = dR_g(\tilde{c}'(0)) \in \mathcal{H}pg$. because dR_g is bijective and the above calculation showed $dR_g\mathcal{H} \subset \mathcal{H}_{pg}$ the second property is also proven □

Pullback of a connection on a Frame Bundle

We saw how a Connection on a Vector Bundles induces a connection on the Frame Bundle. We now explore how a connection on Fr(E) pullsback under a section to give the connection Matrix in the following way.

Theorem

Let ∇ be a connection on a Vector bundle $E \to M$ and ω the Ehrensmann connection induced by the connection.if e_i is a frame on E over an open set U if we view it as a section $e : U \to Fr(E)|_U$ such that for the connection matrix ω_e we have $\omega_e = e^*\omega$

A proof can be found in Loring W. Tu, Differential Geometry □

some Remarks

- Before ending we turn out attention to a final but nontheless crucially important example of a Vector Bundle "associated" to a Principal Bundle.
- In this section we will use tools from Representation theory without proof and refer to: Introduction to representation theory, Pavel Etingof Et. al. for the interested Listener.

the associated bundle

associated bundle

Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to GL(V)$ a representation on some finite dimensional vector space V. we denote$\rho(g)v = gv$ for ease of Notation and define $E := P \times_\rho V$ as the quotient of $P \times V$ by the relation:

$$(p, v) \sim (pg, g^{-1}v) \; for \; g \in G \; and \; (p, v) \in P \times V$$

The associated bundlecomes with a natural projection $\beta : P \times_\rho V \to M$ given by $\beta([p, g]) = \pi(p)$

$$\beta([pg, g^{-1}v]) = \pi(pg) = \pi(p) = \beta([p, v])$$

Fibers of the associated Bundle

We continue without proving that the associated bundle is a Vector Bundle but provide the Maps which will provide the trivialisation.

- $\phi : (U \times G) \times_\rho V \to U \times V$ given by $[(x, g), v] \to (x, gv)$ is well defined and together with $\psi : U \times \to (U \times G) \times_\rho V$ defined by $\psi(x, v) = [(x, 1), v]$ will give a trivialising diffeomorphism.
- The Fibers are the representation Space V
- For any principal G-bundle we can take the adjoint Representation $Ad : G \to GL(\mathfrak{g})$ on it's Lie algebra to define an associated bundle also called an adjoint bundle

Closing Remarks and Outlook

- We saw Fiber bundles+Lie Group$\rightarrow$Prinipal G-bundle.
- We defined a notion of horizontal bundles which gave us a way to define connections.
- We saw how each Vector Bundle naturally admits a Frame Bundle
- furthermore we saw how we can use connections on Vector Bundles to define connections on the frame bundle
- Connections allow us to define curvature on Principal bundles which will be covevered next week.

Thank you for Listening!

frame

Literature used and referenced

- Loring W. Tu, Differential Geometry: Connections, Curvature, and Characteristic Classes. 1st ed., Graduate Texts in Mathematics, vol. 275, Springer, 2017.
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- Frankel, Theodore. The Geometry of Physics: An Introduction. 3rd ed., Cambridge University Press, 2011.